LECTURE NOTE-3
Applied Physics -1

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Gauss' law and its applications

Gauss' law:

Gauss' law define as , ε_0 times the surface integral of the normal component of E over any closed surface in an electrostatic field which equals the net charge inside the surface.

Mathematically, $\oint_s E \,.\, da = \frac{q}{\epsilon_0}$

1. Derivation of the Gauss' law:

Sol^n:

Let us consider, a positive charge q is enclosed in a surface of arbitrary shape as shown in Figure. The electric field **E** at every point of the surface is directed radially outward from the charge, and its magnitude at the point **r** is

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

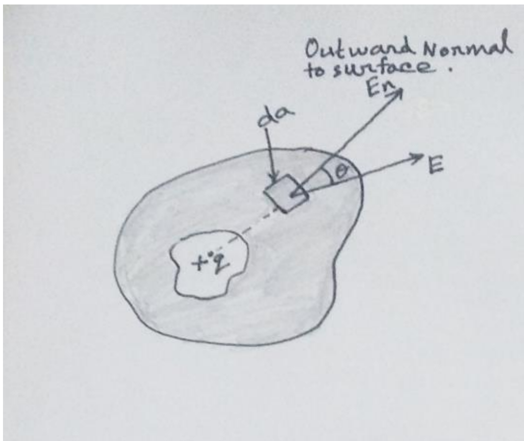

Figure: An imaginary closed surface which encloses a point charge.

Over an infinitesimal area da the magnitude and direction of the field remains the same. The component E along the normal is $E_n = E \cos\theta$, where θ is the angle between E and the outward normal to the surface.

Hence

$$E_n da = E\cos\theta\, da = \frac{q\cos\theta\, da}{4\pi\epsilon_0 r^2} \qquad (1)$$

But $da \cos\theta$ is the projection of da along the radius vector **r** and

$$\frac{da\cos\theta}{r^2} = d\Omega$$

the element of solid angle subtended by da at the point where q is located. Thus

$$E_n da = \frac{q}{4\pi\epsilon_0} d\Omega \qquad (2)$$

Regardless of the shape or size of the closed surface

$$\int d\Omega = 4\pi \text{ Steradians}$$

Hence integrating over the entire closed surface Eq. (2) yields

$$\oint_s E_n da = \frac{q}{4\pi\epsilon_0}\oint d\Omega = \frac{q}{\epsilon_0} \qquad (3)$$

where $\oint_s$ means closed surface integral. Eq. (3) can be put in vector notation if we note that the vector element area da is along the normal. Hence, $E_n da = \mathbf{E} \,.\, \mathbf{da}$. Thus Eq. (3) takes the form

$$\oint_s E \,.\, da = \frac{q}{\epsilon_0}$$

This is the required gauss' Law.

Aplication of Gauss' law:

2. Deduction of Coulomb's law by using Gauss' law:

Sol^n:

Coulomb's law can be deduced from Gauss' law and symmetry considerations. If a point charge q is surrounded by a spherical surface as shown in Figure, then from symmetry consideration the field E is normal to the surface and is constant in magnitude for all point on it. A closed surface so imagined under symmetry consideration will be called a Gaussian surface-a term which we shall often use.

In Figure both E and da at any point of the spherical Gaussian surface have direction radially outward. Hence **E.da** = Eda, and the Gauss' law becomes

$$\oint_s E\,.\,da = \oint_s Eda = \frac{q}{\epsilon_0}$$

But since E is constant on the surface, it can be taken outside the integral sign, and consequently

$$\oint_s E\,.\,da = \oint_s Eda = E\oint_s da = E\,(4\pi r^2) = \frac{q}{\epsilon_0}$$

$$\text{Or,}\quad E = \frac{q}{4\pi\epsilon_0 r^2}$$

In vector form $\overline{E} = \frac{q\,\overline{r}}{4\pi\epsilon_0 r^3}$

We know that, F = qE

Then the force on charge q_o at r is

$$F = \frac{q_0 q}{4\pi\epsilon_0 r^2}$$

In vector form $\overline{F} = \frac{q_0 q}{4\pi\epsilon_0}\frac{\overline{r}}{r^3}$

which is nothing but the Coulomb's law of electrostatic force,

Thus, Coulomb's law can be deduced from Gauss' Law and symmetry considerations.

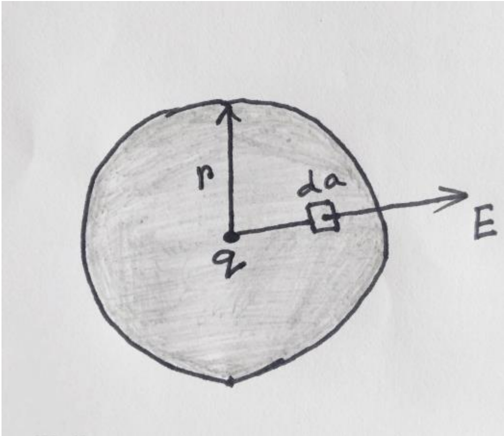

Figure: Spherical Gaussian surface around a point charge.

3. Electric field due to a charged sphere:

Sol^n:

Let us consider a solid conductive sphere of radius R. If an amount of charge q is placed on the sphere it will be distributed uniformly over the surface of the sphere. No charge can reside in the interior region, because it will disturb the normal distribution of charges in this region of the conductor and hence will create an unbalanced electric

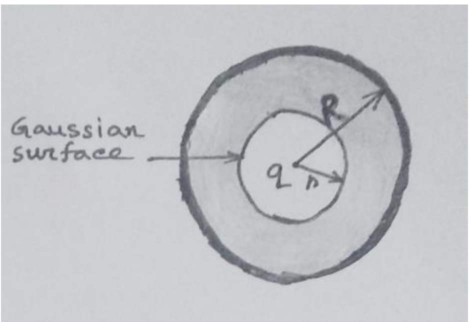

Fig. 1: Gaussian surface inside the charged sphere.

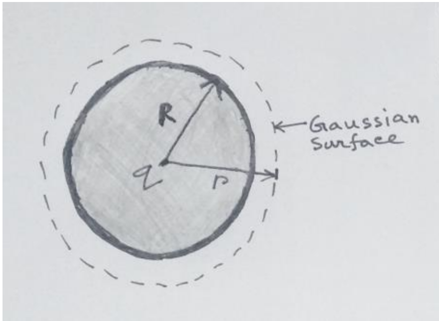

Fig. 2: Gaussian surface outside the charged sphere.

field causing a current to flow. This cannot happen in the case of electrostatic problems. Therefore, the added charge can only reside on the surface of the conductor. But, the distribution of the charge over the surface must be uniform, otherwise there will exist a component of electric field tangential to the surface thus causing a current to flow on the surface. Again this phenomenon cannot happen in the electrostatic case. The distribution of the charge should be such that it would not create a tangential component of the field so that the field is, at all points on the surface, perpendicular. The above discussion is also true for any conductor of arbitrary shape. Since no tangential component of the field exists on the surface of the conductor, the surface will be an equipotential surface.

To find the field at a point $r < R$ of a charged sphere, we imagine a spherical Gaussian surface of radius r concentric with the sphere, **Fig. 1**. From symmetry and the above argument we see that E can be radial and E is uniform over this Gaussian surface which does not contain any charge. So from Gauss' law, $\oint_s E \, . \, da = \frac{q}{\epsilon_0}$

We get $\oint_s E \, . \, da = \oint_s E da = E \oint_s da = E\,(4\pi r^2) = 0$

Hence $E = 0$ for $r < R$

To find the field at $r > R$, we imagine a similar Gaussian surface passing through that point, **Fig. 2**, which now includes the charge q. Again, E is radial and uniform over this Gaussian surface.

So From Gauss' law,

$$\oint_s E \, . \, da = E\,(4\pi r^2) = \frac{q}{\epsilon_0}$$

or, $E = \frac{q}{4\pi\epsilon_0 r^2}$ .... (1)

In vector form $\bar{E} = \frac{q\,\bar{r}}{4\pi\epsilon_0 r^3}$ for $r > R$

Eq. (1) is equivalent to the field at a point r due to a point charge q. Thus, we may say that while finding the electric field at a point outside of a charged sphere, the charge can be considered to be concentrated at the centre of the sphere and we can use the formula for a point charge.

4. Electric field due to a long charged cylinder:

Sol^n: Let us consider, a long cylinder of radius **a** is uniformly charged having a charge λ per unit length. The field E at a point outside the cylinder can be obtained by constructing a Gaussian surface that passes through that point and surrounds an arbitrary length L of the cylinder, shown in Figure. The Gaussian cylindrical surface thus constructed has the symmetry property, and as the charged cylinder is long there is no effect from its ends. The field is everywhere constant on the Gaussian surface, and directed radially away from the axis so that E and da are in the same direction on the curved surface.

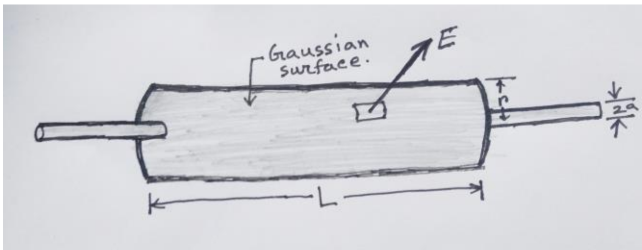

Figure: Cylindrical Gaussian surface for a long charged cylinder

Hence from Guuss' law, $\oint_s E\,.\,da = \frac{q}{\epsilon_0}$ we get

$$\oint_s E\,.\,da = \int_s E\,.\,da = E\,(2\pi r L) = \frac{\lambda}{\epsilon_0} L$$

where integration is taken over the curved surface.

Thus $E = \frac{\lambda}{2\pi\epsilon_0 r}$

In vector notation $\bar{E} = \frac{\lambda \bar{r}}{2\pi\epsilon_0 r^2}$ for $r > a$

The charged cylinder behaves as if the charge is distributed on a thin wire which passes along the axis of the wire.

To find the field in the interior of the charged cylinder, we note that no charge can reside in this region and hence applying Gauss' law we find E = 0 for r < a.

5. Electric field due to a uniformly chargea plane:

Sol^n: Let us consider, a uniformly charged plane of infinite extent having charged σ per unit area of the surface. An area ΔA of the plane can be enclosed by a Gaussian cylinder of the same cross-section as shown in Figure. Now, E is uniform, parallel to da on the ends and perpendicular to da on the curved surface. Hence, we obtain from Gauss' Law

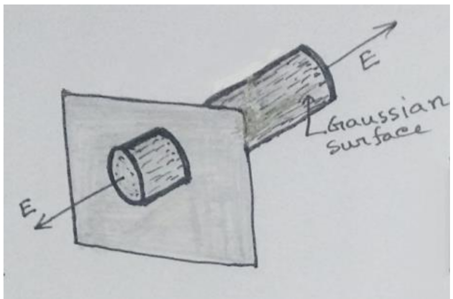

Figure: Cylindrical Gaussian surface for a large uniformly charged plane.

$$\oint_s E\,.\,da = \frac{q}{\epsilon_0} = \frac{1}{\epsilon_0}\int_s \sigma\, da$$

or, $E\Delta A + E\Delta A = \frac{1}{\epsilon_0}\sigma\Delta A$

Hence $E = \frac{\sigma}{2\epsilon_0}$

This is the required expression for the Electric field due to a uniformly charged plane.

6. Field due to two parallel charged plates:

Sol^n:

Let AB and CD be two parallel plates of very great extent. AB has positive charge and CD has negative. From figure we get, at point P, the field due to plate AB is $E_1 = \frac{\sigma}{2\epsilon_0}$, pointing right, and that due to plate CD is $E_2 = \frac{\sigma}{2\epsilon_0}$, pointing right

(because the plate CD is charged negatively and E must direct towards it). Therefore, the total field at P is, by the principle of superposition,

$$E = E_1 + E_2 = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}$$

the direction being from plate AB to plate CD, that is, from positive charge to negative charge.

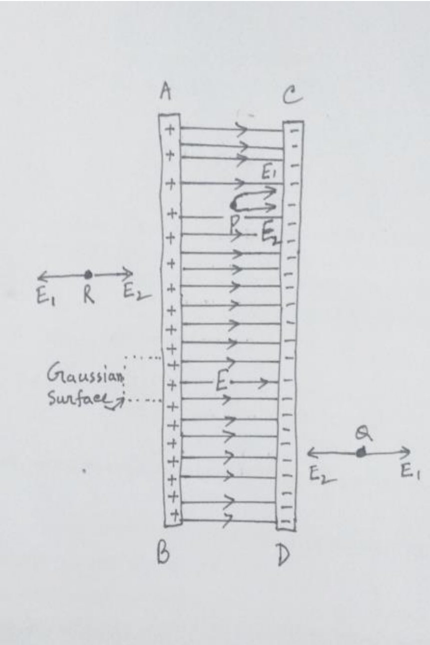

Figure: Gaussian surfaces for two parallel charged plates.

The field at Q however, is $E_1 = \frac{\sigma}{2\epsilon_0}$, towards right due to plate AB and $E_2 = \frac{\sigma}{2\epsilon_0}$, towards left due to plate CD and the total field is zero. This proves that the field at points outside the plates, such as Q and R, vanishes.